

Chapter 2

Electric Fields

2.1 The Important Stuff

2.1.1 The Electric Field

Suppose we have a point charge q_0 located at \mathbf{r} and a set of *external* charges conspire so as to exert a force \mathbf{F} on this charge. We can define the **electric field** at the point \mathbf{r} by:

$$\mathbf{E} = \frac{\mathbf{F}}{q_0} \quad (2.1)$$

The (vector) value of the \mathbf{E} field depends *only* on the values and locations of the external charges, because from Coulomb's law the force on any "test charge" q_0 is proportional to the value of the charge. However to make this definition really kosher we have to stipulate that the test charge q_0 is "small"; otherwise its presence will significantly influence the locations of the external charges.

Turning Eq. 2.1 around, we can say that if the electric field at some point \mathbf{r} has the value \mathbf{E} then a *small* charge placed at \mathbf{r} will experience a force

$$\mathbf{F} = q_0 \mathbf{E} \quad (2.2)$$

The electric field is a *vector*. From Eq. 2.1 we can see that its SI units must be $\frac{\text{N}}{\text{C}}$.

It follows from Coulomb's law that the electric field at point \mathbf{r} due to a charge q located at the origin is given by

$$\mathbf{E} = k \frac{q}{r^2} \hat{\mathbf{r}} \quad (2.3)$$

where $\hat{\mathbf{r}}$ is the unit vector which points in the same direction as \mathbf{r} .

2.1.2 Electric Fields from Particular Charge Distributions

- **Electric Dipole**

An **electric dipole** is a pair of charges of opposite sign ($\pm q$) separated by a distance d which is usually meant to be small compared to the distance from the charges at which we

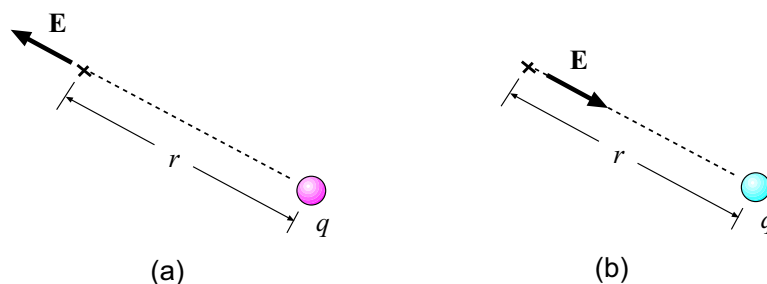


Figure 2.1: The E field due to a point charge q . (a) If the charge q is positive, the E field at some point a distance r away has magnitude $k|q|/r^2$ and points *away* from the charge. (b) If the charge q is negative, the E field has magnitude $k|q|/r^2$ and points *toward* the charge.

want to find the electric field. The product qd turns out to be important; the vector which points from the $-q$ charge to the $+q$ charge and has magnitude qd is known as the **electric dipole moment** for the pair, and is denoted \mathbf{p} .

Suppose we form an electric dipole by placing a charge $+q$ at $(0, 0, d/2)$ and a charge $-q$ at $(0, 0, -d/2)$. (So the dipole moment \mathbf{p} has magnitude $p = qd$ and points in the $+\mathbf{k}$ direction.) One can show that when z is much larger than d , the electric field for points on the z axis is

$$E_z = \frac{1}{2\pi\epsilon_0} \frac{p}{z^3} = k \frac{2qd}{z^3} \quad (2.4)$$

• “Line” of Charge

A linear charge distribution is characterized by its charge per unit length. **Linear charge density** is usually given the symbol λ ; for an arclength ds of the distribution, the electric charge is

$$dq = \lambda ds$$

For a ring of charge with radius R and total charge q , for a point on the axis of the ring a distance z from the center, the magnitude of the electric field (which points along the z axis) is

$$E = \frac{qz}{4\pi\epsilon_0(z^2 + R^2)^{3/2}} \quad (2.5)$$

• Charged Disk & Infinite Sheet

A two-dimensional (surface) distribution of charge is characterized by its charge per unit area. **Surface charge density** is usually given the symbol σ ; for an area element dA of the distribution, the electric charge is

$$dq = \sigma dA$$

For a disk of radius R and uniform charge density σ on its surface, for a point on the axis of the disk at a distance z away from the center, the magnitude of the electric field (which points along the z axis) is

$$E = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \quad (2.6)$$

The limit $R \rightarrow \infty$ of Eq. 2.6 gives the magnitude of the E field at a distance z from an infinite sheet of charge with charge density σ . The result is

$$E = \frac{\sigma}{2\epsilon_0} \quad (2.7)$$

2.1.3 Forces on Charges in Electric Fields

An isolated charge q in an electric field experiences a force $\mathbf{F} = q\mathbf{E}$. We note that when q is positive the force points in the same direction as the field, but when q is negative, the force is opposite the field direction!

The potential energy of a point charge in an \mathbf{E} field will be discussed at great length in chapter 4!

When an electric dipole \mathbf{p} is placed in a uniform \mathbf{E} field, it experiences no net force, but it *does* experience a torque. The torque is given by:

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} \quad (2.8)$$

The potential energy of a dipole also depends on its orientation, and is given by:

$$U = -\mathbf{p} \cdot \mathbf{E} \quad (2.9)$$

2.1.4 Electric Field Lines

Oftentimes it is useful for us to get an *overall visual picture* of the electric field due to a particular distribution of charge. It is useful to make a plot where the little arrows representing the direction of the electric field at each point are joined together, forming continuous (directed) “lines”. These are the **electric field lines** for the charge distribution.

Such a plot will tell us the basic *direction* of the electric field at all points in space (though we do lose information about the *magnitude* of the field when we join the arrows). One can show that:

- Electric field lines originate on positive charges (they point *away* from the positive charge) and end on negative charges (they point *toward* the negative charge).
- Field lines cannot cross one another.

Whereas a diagram of field lines can contain as many lines as you please, for an accurate representation of the field the number of lines originating from a charge should be *proportional* to the charge.

2.2 Worked Examples

2.2.1 The Electric Field

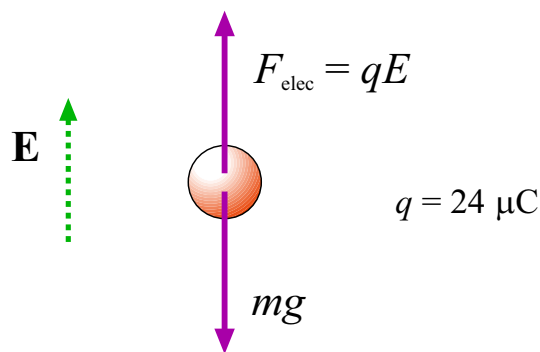


Figure 2.2: Forces acting on the charged mass in Example 1.

1. An object having a net charge of $24\ \mu\text{C}$ is placed in a uniform electric field of $610\ \frac{\text{N}}{\text{C}}$ directed vertically. What is the mass of this object if it “floats” in the field? [Ser4 23-16]

The forces acting on the mass are shown in Fig. 2.2. The force of gravity points downward and has magnitude mg (m is the mass of the object) and the electrical force acting on the mass has magnitude $F = |q|E$, where q is the charge of the object and E is the magnitude of the electric field. The object “floats”, so the net force is zero. This gives us:

$$|q|E = mg$$

Solve for m :

$$m = \frac{|q|E}{g} = \frac{(24 \times 10^{-6}\ \text{C})(610\ \frac{\text{N}}{\text{C}})}{(9.80\ \frac{\text{m}}{\text{s}^2})} = 1.5 \times 10^{-3}\ \text{kg}$$

The mass of the object is $1.5 \times 10^{-3}\ \text{kg} = 1.5\ \text{g}$.

2. An electron is released from rest in a uniform electric of magnitude $2.00 \times 10^4\ \frac{\text{N}}{\text{C}}$. Calculate the acceleration of the electron. (Ignore gravitation.) [HRW6 23-29]

The magnitude of the force on a charge q in an electric field is given by $F = |qE|$, where E is the magnitude of the field. The magnitude of the electron’s charge is $e = 1.602 \times 10^{-19}\ \text{C}$, so the magnitude of the force on the electron is

$$F = |qE| = (1.602 \times 10^{-19}\ \text{C})(2.00 \times 10^4\ \frac{\text{N}}{\text{C}}) = 3.20 \times 10^{-15}\ \text{N}$$

Newton’s 2nd law relates the magnitudes of the force and acceleration: $F = ma$, so the acceleration of the electron has magnitude

$$a = \frac{F}{m} = \frac{(3.20 \times 10^{-15}\ \text{N})}{(9.11 \times 10^{-31}\ \text{kg})} = 3.51 \times 10^{15}\ \frac{\text{m}}{\text{s}^2}$$

That’s the *magnitude* of the electron’s acceleration. Since the electron has a negative charge the direction of the *force* on the electron (and also the acceleration) is *opposite* the direction of the electric field.

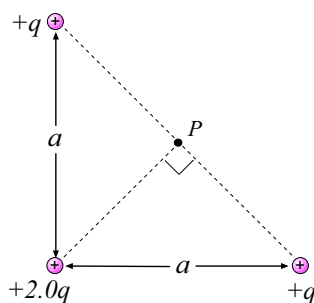


Figure 2.3: Charge configuration for Example 4.

3. What is the magnitude of a point charge that would create an electric field of $1.00 \frac{\text{N}}{\text{C}}$ at points 1.00 m away? [HRW6 23-4]

From Eq. 2.3, the magnitude of the E field due to a point charge q at a distance r is given by

$$E = k \frac{|q|}{r^2}$$

Here we are given E and r , so we can solve for $|q|$:

$$\begin{aligned} |q| &= \frac{Er^2}{k} = \frac{(1.00 \frac{\text{N}}{\text{C}})(1.00 \text{ m})^2}{(8.99 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2})} \\ &= 1.11 \times 10^{-10} \text{ C} \end{aligned}$$

The *magnitude* of the charge is $1.11 \times 10^{-10} \text{ C}$.

4. Calculate the direction and magnitude of the electric field at point P in Fig. 2.3, due to the three point charges. [HRW6 23-12]

Since each of the three charges is *positive* they give electric fields at P pointing *away* from the charges. This is shown in Fig. 2.4, where the charges are individually numbered along with their (vector!) E -field contributions.

We note that charges 1 and 2 have the same magnitude and are both at the same distance from P . So the E -field vectors for these charges shown in Fig. 2.4 (being in opposite directions) must cancel. So we are left with only the contribution from charge 3.

We know the direction for this vector; it is 45° above the x axis. To find its magnitude we note that the distance of this charge from P is half the length of the square's diagonal, or:

$$r = \frac{1}{2}(\sqrt{2}a) = \frac{a}{\sqrt{2}}$$

and so the magnitude is

$$E_3 = k \frac{2q}{r^2} = \frac{2kq}{(a/\sqrt{2})^2} = \frac{4kq}{a^2} .$$

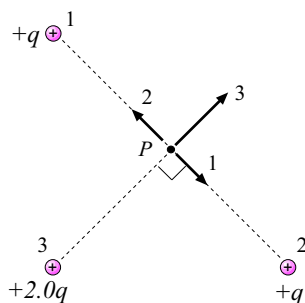


Figure 2.4: Directions for the contributions to the E field at P due to the three positive charges in Example 4.

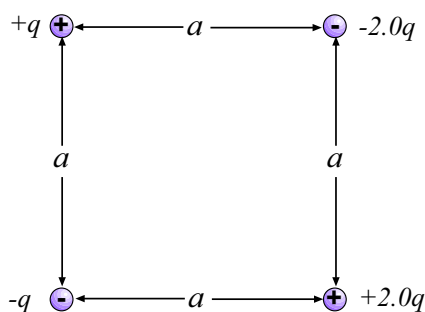


Figure 2.5: Charge configuration for Example 5.

So the electric field at P has magnitude

$$E_{\text{net}} = \frac{4kq}{a^2} = \frac{4q}{(4\pi\epsilon_0)a^2} = \frac{q}{\pi\epsilon_0 a^2}$$

and points at an angle of 45° .

5. What are the magnitude and direction of the electric field at the center of the square of Fig. 2.5 if $q = 1.0 \times 10^{-8} \text{ C}$ and $a = 5.0 \text{ cm}$? [HRW6 23-13]

The center of the square is equidistant from all the charges. This distance r is half the diagonal of the square, hence

$$r = \frac{1}{2}(\sqrt{2}a) = \frac{a}{\sqrt{2}} = \frac{(5.0 \text{ cm})}{\sqrt{2}} = 3.55 \times 10^{-2} \text{ m}$$

Then we can find the *magnitudes* of the contributions to the \mathbf{E} field from each of the charges. The charges of magnitude q have contributions of magnitude

$$E_{1.0q} = k \frac{q}{r^2} = (8.99 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}) \frac{(1.0 \times 10^{-8} \text{ C})}{(3.55 \times 10^{-2} \text{ m})^2} = 7.13 \times 10^4 \frac{\text{N}}{\text{C}}$$

The charges of magnitude $2.0q$ contribute with fields of twice this magnitude, namely

$$E_{2.0q} = 2E_{1.0q} = 1.43 \times 10^5 \frac{\text{N}}{\text{C}}$$

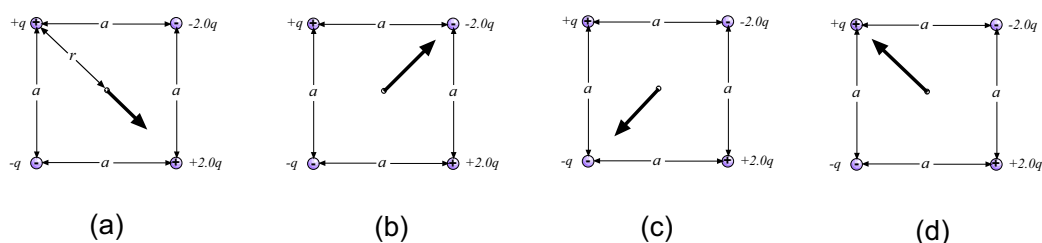


Figure 2.6: Directions of \mathbf{E} field at the center of the square due to three of the corner charges. (a) Upper left charge is at distance $r = a/\sqrt{2}$ from the center (as are the other charges). \mathbf{E} field due to this charge points *away from* charge, in -45° direction. (b) \mathbf{E} field due to upper right charge points *toward* charge, in $+45^\circ$ direction. (c) \mathbf{E} field due to lower left charge points *toward* charge, in $+225^\circ$ direction. (d) \mathbf{E} field due to lower right charge points *away from* charge, in $+135^\circ$ direction.

The *directions* of the contributions to the total \mathbf{E} field are shown in Fig. 2.6(a)–(d). The \mathbf{E} field due to the upper left charge points *away from* charge, which is in -45° direction (as measured from the $+x$ axis, as usual). The \mathbf{E} field due to upper right charge points *toward* the charge, in $+45^\circ$ direction. The \mathbf{E} field due to lower left charge points *toward* that charge, in $180^\circ + 45^\circ = +225^\circ$ direction. Finally, \mathbf{E} field due to lower right charge points *away from* charge, in $180^\circ - 45^\circ = +135^\circ$ direction.

So we now have the magnitudes and directions of four vectors. Can we add them together? Sure we can!

$$\begin{aligned}\mathbf{E}_{\text{Total}} &= (7.13 \times 10^4 \frac{\text{N}}{\text{C}})(\cos(-45^\circ)\mathbf{i} + \sin(-45^\circ)\mathbf{j}) \\ &+ (1.43 \times 10^5 \frac{\text{N}}{\text{C}})(\cos(+45^\circ)\mathbf{i} + \sin(45^\circ)\mathbf{j}) \\ &+ (7.13 \times 10^4 \frac{\text{N}}{\text{C}})(\cos(225^\circ)\mathbf{i} + \sin(225^\circ)\mathbf{j}) \\ &+ (1.43 \times 10^5 \frac{\text{N}}{\text{C}})(\cos(+135^\circ)\mathbf{i} + \sin(135^\circ)\mathbf{j})\end{aligned}$$

(I know, this is the clumsy way of doing it, but I'll get to that.) The sum gives:

$$\mathbf{E}_{\text{Total}} = 0.0\mathbf{i} + (1.02 \times 10^5 \frac{\text{N}}{\text{C}})\mathbf{j}$$

So the magnitude of $\mathbf{E}_{\text{Total}}$ is $1.02 \times 10^5 \frac{\text{N}}{\text{C}}$ and it points in the $+y$ direction.

This particular problem can be made easier by noting the cancellation of the \mathbf{E} 's contributed by the charges on opposite corners of the square. For example, a $+q$ charge in the upper left and a $+2.0q$ charge in the lower right is equivalent to a *single* charge $+q$ in the lower right (as far as this problem is concerned).

2.2.2 Electric Fields from Particular Charge Distributions

6. Electric Quadrupole Fig. 2.7 shows an electric quadrupole. It consists of two dipole moments that are equal in magnitude but opposite in direction. Show that the value of E on the axis of the quadrupole for points a distance z from its center (assume $z \gg d$) is given by

$$E = \frac{3Q}{4\pi\epsilon_0 z^4},$$

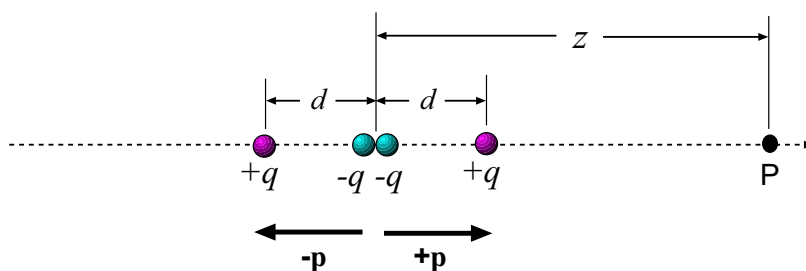


Figure 2.7: Charges forming electric quadrupole in Example 6.

in which Q (defined by $Q \equiv 2qd^2$) is known as the *quadrupole moment of the charge distribution*. [HRW6 23-17]

We note that as the problem is given we really have *three* separate charges in this configuration: A charge $-2q$ at the origin, a charge $+q$ at $z = -d$ and a charge $+q$ at $z = +d$. (Again, see Fig. 2.7.)

We are assuming that q is positive; for now let us also assume that the point P (for which we want the electric field) is located on the z axis at some positive value of z , as indicated in Fig. 2.7. We will now find the contribution to the electric field at P for each of the three charges.

The center charge ($-2q$) lies at a distance z from the point P . So then the *magnitude* of the E field due to this charge is $k\frac{2q}{z^2}$, but since the charge is *negative* the field points *toward* the charge, which in this case is in the $-z$ direction. So then the contribution E_z (at point P) by the center charge is

$$E_z^{(\text{center})} = -k\frac{2q}{z^2}$$

The charge on the left ($+q$) lies at a distance $z + d$ from the point P . So then the magnitude of the E field due to this charge is $k\frac{q}{(z+d)^2}$. Since this charge is positive, this field points *away from* the charge, namely in the $+z$ direction. So the contribution to E_z by the left charge is

$$E_z^{(\text{left})} = +k\frac{q}{(z+d)^2}$$

The charge on the right ($+q$) lies at a distance $z - d$ from the point P . The magnitude of the field due to this charge is $k\frac{q}{(z-d)^2}$ and this charge is also positive so we get a contribution

$$E_z^{(\text{right})} = +k\frac{q}{(z-d)^2}$$

The field at point P is the sum of the three contributions. We factor out kq from each term to get:

$$E_z = kq \left[-\frac{2}{z^2} + \frac{1}{(z+d)^2} + \frac{1}{(z-d)^2} \right].$$

At this point we are really done with the *physics* of the problem; the rest of the work is doing the *mathematical* steps to get a simpler (approximate) expression for E_z .

We can remove a factor of z^2 from the denominator of each term. We choose z because it is much larger than d and later this will allow us to use the binomial expansion. We get:

$$\begin{aligned}
 E_z &= kq \left[-\frac{2}{z^2} + \frac{1}{z^2 \left(1 + \frac{d}{z}\right)^2} + \frac{1}{z^2 \left(1 - \frac{d}{z}\right)^2} \right] \\
 &= \frac{kq}{z^2} \left[-2 + \frac{1}{\left(1 + \frac{d}{z}\right)^2} + \frac{1}{\left(1 - \frac{d}{z}\right)^2} \right] \\
 &= \frac{kq}{z^2} \left[-2 + \left(1 + \frac{d}{z}\right)^{-2} + \left(1 - \frac{d}{z}\right)^{-2} \right] \tag{2.10}
 \end{aligned}$$

At this point the expression for E_z is exact, but it is useful to get an approximate expression for the case where z is much larger than the size of the quadrupole: $z \gg d$. For this we can make use of the binomial expansion (another name for the Taylor expansion of $(1+x)^n$ about $x=0$):

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots \quad \text{valid for } x^2 < 1$$

The formula is especially useful when x is small in absolute value; then the first several terms of the expansion will give a good approximation. We will use the binomial expansion to simplify our last expression by associating $\pm \frac{d}{z}$ with x (because it *is* small), using $n = -2$, and just to be safe, we'll use the first three terms. This gives us the approximations:

$$\begin{aligned}
 \left(1 + \frac{d}{z}\right)^{-2} &\approx 1 - 2\left(\frac{d}{z}\right) + \frac{6}{2}\left(\frac{d}{z}\right)^2 = 1 - \frac{2d}{z} + 3\left(\frac{d}{z}\right)^2 \\
 \left(1 - \frac{d}{z}\right)^{-2} &\approx 1 + 2\left(\frac{d}{z}\right) + \frac{6}{2}\left(\frac{d}{z}\right)^2 = 1 + \frac{2d}{z} + 3\left(\frac{d}{z}\right)^2
 \end{aligned}$$

Now putting these results into Eq. 2.10 we find:

$$\begin{aligned}
 E_z &\approx \frac{kq}{z^2} \left[-2 + 1 - \frac{2d}{z} + 3\left(\frac{d}{z}\right)^2 + 1 + \frac{2d}{z} + 3\left(\frac{d}{z}\right)^2 \right] \\
 &= \frac{kq}{z^2} \left[6\left(\frac{d}{z}\right)^2 \right] \\
 &= \frac{6kqd^2}{z^4}
 \end{aligned}$$

Using the definition $Q \equiv 2qd^2$, and also $k = \frac{1}{4\pi\epsilon_0}$ we can also write this result as

$$E_z = \frac{3kQ}{z^4} = \frac{3Q}{4\pi\epsilon_0 z^4}.$$

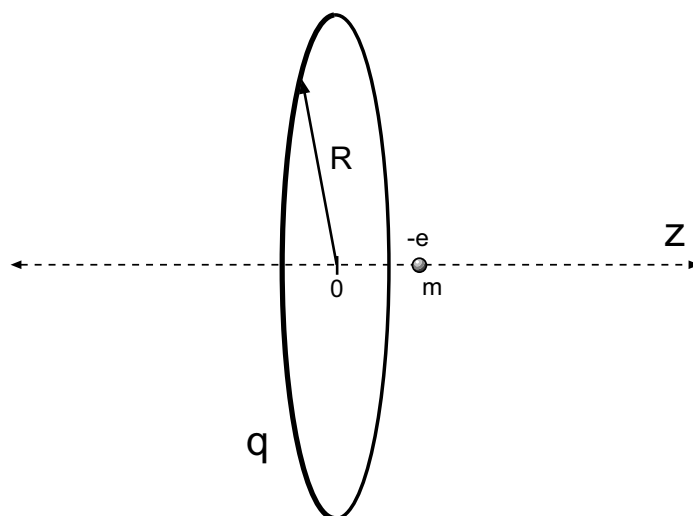


Figure 2.8: Electron oscillates on z axis through center of charged ring of radius R and total charge q , as in Example 7.

7. An electron is constrained to the central axis of the ring of charge with radius R and total charge q . Show that the electrostatic force exerted on the electron can cause it to oscillate through the center of the ring with an angular frequency

$$\omega = \sqrt{\frac{eq}{4\pi\epsilon_0 m R^3}},$$

where m is the electron's mass. [HRW6 23-19]

A picture of this problem is given in Fig. 2.8; included is the electron which oscillates through the center. For this to happen, the charge q must be positive so that the electron is always attracted back to the ring (i.e. the force is *restoring*.)

From Eq. 2.5 we have the magnitude of the E field for points on the axis of the ring (which we will call the z axis, with its origin at the center of the ring):

$$E = \frac{q|z|}{4\pi\epsilon_0(z^2 + R^2)^{3/2}}$$

Note, we need an absolute value sign on the coordinate z to get a *positive* magnitude. Now, the E field points along the $\pm z$ axis; when z is positive it goes in the $+z$ direction and when z is negative it goes in the $-z$ direction. So the z component of the electric field which the electron “sees” at coordinate z is in fact

$$E_z = \frac{qz}{4\pi\epsilon_0(z^2 + R^2)^{3/2}}$$

From Eq. 2.2 we get force on the electron as it moves on the z axis:

$$F_z = (-e)E_z = \frac{-eqz}{4\pi\epsilon_0(z^2 + R^2)^{3/2}} \quad (2.11)$$

This expression for the force on the electron is rather messy; it is a *restoring* force since its direction is opposite the displacement of the electron from the center, but it is not *linear*, that is, it is not simply proportional to z . (Our work on harmonic motion assumed a *linear* restoring force.) We will assume that the oscillations are *small* in the sense that the maximum value of $|z|$ is much smaller than R . If that is true, then in the denominator of Eq. 2.11 we can approximate:

$$(z^2 + R^2)^{3/2} \approx (R^2)^{3/2} = R^3$$

Making this replacement in Eq. 2.11, we find:

$$F_z \approx \frac{-eqz}{4\pi\epsilon_0 R^3} = -\left(\frac{eq}{4\pi\epsilon_0 R^3}\right)z$$

This is a force “law” which just like the Hooke’s Law, $F_z = -kz$ with the role of the spring constant k being played by

$$k \Leftrightarrow \frac{eq}{4\pi\epsilon_0 R^3} \quad (2.12)$$

By analogy with the harmonic motion of a mass m on the end of a spring of force constant k , where the result for the angular frequency was $\omega = \sqrt{\frac{k}{m}}$, using Eq. 2.12 we find that angular frequency for the electron’s motion is

$$\omega = \sqrt{\frac{eq}{4\pi\epsilon_0 R^3 m}}$$

8. A thin nonconducting rod of finite length L has a charge q spread uniformly along it. Show that the magnitude E of the electric field at point P on the perpendicular bisector of the rod (see Fig. 2.9) is given by

$$E = \frac{q}{2\pi\epsilon_0 y} \frac{1}{(L^2 + 4y^2)^{1/2}} .$$

[HRW6 23-24]

We first set up a coordinate system with which to do our calculation. Let the origin be at the center of the rod and let the x axis extend along the rod. In this system, the point P is located at $(0, y)$ and the ends of the rod are at $(-L/2, 0)$ and $(+L/2, 0)$.

If the charge q is spread uniformly over the rod, then it has a linear charge density of

$$\lambda = \frac{q}{L} .$$

Then if we take a section of the rod of length dx , it will contain a charge λdx .

Next, we consider how a *tiny bit* of the rod will contribute to the electric field at P . This is shown in Fig. 2.10. We consider a piece of the rod of length dx , centered at the coordinate

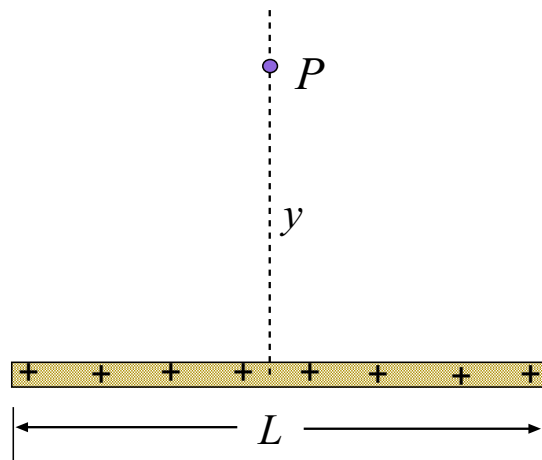
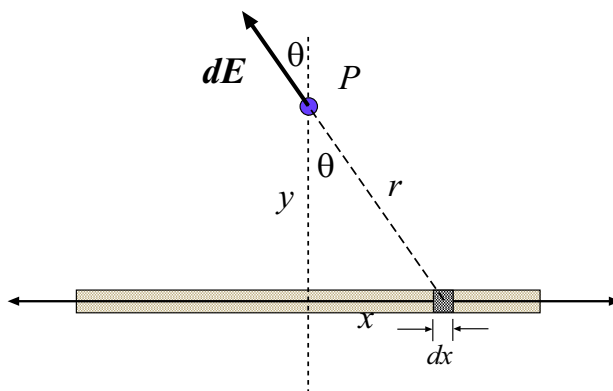


Figure 2.9: Charged rod and geometry for Example 8.

Figure 2.10: An element of the rod of length dx located at x gives a contribution to the electric field at point P .

x . This element is small enough that we can treat it as a point charge... and we *know* how to find the electric field due to a point charge.

The distance of this small piece from the point P is

$$r = \sqrt{x^2 + y^2}$$

and the amount of electric charge contained in the piece is

$$dq = \lambda dx$$

Therefore the electric charge in this little bit of the rod gives an electric field of magnitude

$$dE = k \frac{dq}{r^2} = \frac{k\lambda dx}{(x^2 + y^2)}$$

This little bit of electric field $d\mathbf{E}$ points at an angle θ away from the y axis, as shown in Fig. 2.10.

Eventually we will have to *add up* all the little bits of electric field $d\mathbf{E}$ due to each little bit of the rod. In doing so we will be adding *vectors* and so we will need the components of the $d\mathbf{E}$'s. As can be seen from Fig. 2.10, the components are:

$$dE_x = dE \sin \theta = -\frac{k\lambda dx}{(x^2 + y^2)} \sin \theta \quad \text{and} \quad dE_y = \cos \theta dE = \frac{k\lambda dx}{(x^2 + y^2)} \cos \theta \quad (2.13)$$

Also, some basic trigonometry gives us:

$$\sin \theta = \frac{x}{r} = \frac{x}{(x^2 + y^2)^{1/2}} \quad \cos \theta = \frac{y}{r} = \frac{y}{(x^2 + y^2)^{1/2}}$$

Using these in Eqs. 2.13 gives:

$$dE_x = -\frac{kx\lambda dx}{(x^2 + y^2)^{3/2}} \quad dE_y = \frac{ky\lambda dx}{(x^2 + y^2)^{3/2}} \quad (2.14)$$

The next step is to add up all the individual dE_x 's and dE_y 's. The result for the sum of the dE_x 's is easy: It must be zero! By considering *all* of the little bits of the rod, we can see that dE_x will be positive just as often as it is negative, and when the sum is taken the result is zero. (We sometimes say that this result follows "from symmetry".) The same is not true for the dE_y 's; they are always positive and we *will* have to do some work to add them up.

Eq. 2.14 gives the contribution to E_y arising from an element of length dx centered on x . The bits of the rod extend from $x = -L/2$ to $x = +L/2$, so to get the sum of *all* the little bits we do the integral:

$$E_y = \int_{\text{rod}} dE_y = \int_{-L/2}^{+L/2} \frac{ky\lambda dx}{(x^2 + y^2)^{3/2}} \quad (2.15)$$

Eq. 2.15 gives the result for the E field at P (which is what the problem asks for) so we are now done with the *physics* of the problem. All that remains is some *mathematics* to work out the integral in Eq. 2.15.

First, since k , λ and y are constants as far as the integral is concerned, they can be taken outside the integral sign:

$$E_y = k\lambda y \int_{-L/2}^{+L/2} \frac{1}{(x^2 + y^2)^{3/2}} dx$$

The integral here is not difficult; it can be looked up in a table or evaluated by computer. We find:

$$E_y = k\lambda y \frac{x}{y^2 \sqrt{x^2 + y^2}} \Big|_{-L/2}^{+L/2}$$

Evaluate!

$$\begin{aligned} E_y &= k\lambda y \left\{ \frac{L/2}{y^2 \left(\frac{L^2}{4} + y^2\right)^{1/2}} - \frac{(-L/2)}{y^2 \left(\frac{L^2}{4} + y^2\right)^{1/2}} \right\} \\ &= k\lambda y \frac{L}{y^2 \left(\frac{L^2}{4} + y^2\right)^{1/2}} \end{aligned}$$

We can cancel a factor of y ; also, make the replacements $k = \frac{1}{4\pi\epsilon_0}$ and $\lambda = q/L$. This gives:

$$\begin{aligned} E_y &= \frac{1}{4\pi\epsilon_0} \frac{q}{L} \frac{L}{y \left(\frac{L^2}{4} + y^2\right)^{1/2}} \\ &= \frac{q}{4\pi\epsilon_0} \frac{1}{y \left(\frac{L^2}{4} + y^2\right)^{1/2}} \end{aligned}$$

We're getting close! We can make the expression look a little neater by pulling a factor of $\frac{1}{4}$ out of the parentheses in the denominator. Use:

$$\left(\frac{L^2}{4} + y^2\right)^{1/2} = \frac{1}{2} (L^2 + 4y^2)^{1/2}$$

in our last expression and finally get:

$$\begin{aligned} E_y &= \frac{q}{4\pi\epsilon_0} \frac{2}{y (L^2 + 4y^2)^{1/2}} \\ &= \frac{q}{2\pi\epsilon_0 y} \frac{1}{(L^2 + 4y^2)^{1/2}} \end{aligned}$$

Since at point P , \mathbf{E} has no x component, the magnitude of the E field is

$$E = \frac{q}{2\pi\epsilon_0 y} \frac{1}{(L^2 + 4y^2)^{1/2}}$$

(and the field points in the $+y$ direction for positive q .)

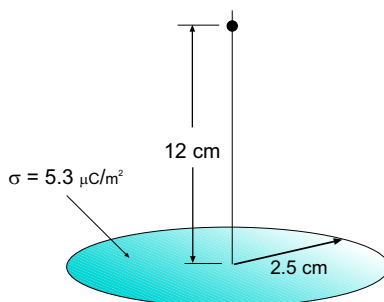


Figure 2.11: Geometry for Example 9.

9. A disk of radius 2.5 cm has a surface charge density of $5.3 \frac{\mu\text{C}}{\text{m}^2}$ on its upper face. What is the magnitude of the electric field produced by the disk at a point on its central axis at distance $z = 12$ cm from the disk? [HRW6 23-26]

The geometry for this problem is shown in Fig. 2.11. Here we *do* have a formula we can use, Eq. 2.6. With $r = 2.5 \times 10^{-2}$ m, $\sigma = 5.3 \frac{\mu\text{C}}{\text{m}^2}$ and $z = 12 \times 10^{-2}$ m we find:

$$\begin{aligned} E &= \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + r^2}} \right) \\ &= \frac{(5.3 \frac{\mu\text{C}}{\text{m}^2})}{2 \left(8.85 \times 10^{-12} \frac{\text{C}^2}{\text{N}\cdot\text{m}^2} \right)} \left(1 - \frac{(12 \times 10^{-2} \text{ m})}{\sqrt{(12 \times 10^{-2} \text{ m})^2 + (2.5 \times 10^{-2} \text{ m})^2}} \right) \\ &= 629 \frac{\text{N}}{\text{C}} \end{aligned}$$

At the given point, the E field has magnitude $629 \frac{\text{N}}{\text{C}}$ and points away from the disk.

2.2.3 Forces on Charges in Electric Fields

10. An electron and a proton are each placed at rest in an electric field of 520 N/C. Calculate the speed of each particle 48 ns after being released. [Ser4 23-44]

Consider the electron. From $\mathbf{F} = q\mathbf{E}$, and the fact that the *magnitude* of the electron's charge is 1.60×10^{-19} C, the *magnitude* of the force on the electron is

$$F = |q|E = (1.60 \times 10^{-19} \text{ C})(520 \text{ N/C}) = 8.32 \times 10^{-17} \text{ N}$$

and since the mass of the electron is $m_e = 9.11 \times 10^{-31}$ kg, from Newton's 2nd Law, the *magnitude* of its acceleration is

$$a = \frac{F}{m_e} = \frac{(8.32 \times 10^{-17} \text{ N})}{(9.11 \times 10^{-31} \text{ kg})} = 9.13 \times 10^{13} \frac{\text{m}}{\text{s}^2}$$

Since the electron starts from rest ($\mathbf{v}_0 = 0$), we have $\mathbf{v} = \mathbf{a}t$ and so the *magnitude* of its velocity 48 ns after being released is

$$v = at = (9.13 \times 10^{13} \frac{\text{m}}{\text{s}^2})(48 \times 10^{-9} \text{ s}) = 4.4 \times 10^6 \frac{\text{m}}{\text{s}} .$$

So the final *speed* of the electron is $4.4 \times 10^6 \frac{\text{m}}{\text{s}}$.

We do a similar calculation for the proton; the only difference is its larger mass. Since the *magnitude* of the proton's charge is the same as that of the electron, the magnitude of the force will be the same:

$$F = 8.32 \times 10^{-17} \text{ N}$$

But as the proton mass is $1.67 \times 10^{-27} \text{ kg}$, its acceleration has magnitude

$$a = \frac{F}{m} = \frac{(8.32 \times 10^{-17} \text{ N})}{(1.67 \times 10^{-27} \text{ kg})} = 4.98 \times 10^{10} \frac{\text{m}}{\text{s}^2}$$

And then the magnitude of the velocity 48 ns after being released is

$$v = at = (4.98 \times 10^{10} \frac{\text{m}}{\text{s}^2})(48 \times 10^{-9} \text{ s}) = 2.4 \times 10^3 \frac{\text{m}}{\text{s}} .$$

So the proton's final speed is $2.4 \times 10^3 \frac{\text{m}}{\text{s}}$.

11. Beams of high-speed protons can be produced in “guns” using electric fields to accelerate the protons. (a) What acceleration would a proton experience if the gun's electric field were $2.00 \times 10^4 \frac{\text{N}}{\text{C}}$? (b) What speed would the proton attain if the field accelerated the proton through a distance of 1.00 cm? [HRW6 23-35]

(a) The proton has charge $+e = 1.60 \times 10^{-19} \text{ C}$ so in the given (uniform) electric field, the force on the protons has magnitude

$$F = |q|E = eE = (1.60 \times 10^{-19} \text{ C})(2.00 \times 10^4 \frac{\text{N}}{\text{C}}) = 3.20 \times 10^{-15} \text{ N}$$

Then we use Newton's second law to get the magnitude of the protons' acceleration. Using the mass of the proton, $m_p = 1.67 \times 10^{-27} \text{ kg}$,

$$a = \frac{F}{m_p} = \frac{(3.20 \times 10^{-15} \text{ N})}{(1.67 \times 10^{-27} \text{ kg})} = 1.92 \times 10^{12} \frac{\text{m}}{\text{s}^2}$$

(b) The protons start from rest (this is assumed) and move in one dimension, accelerating with $a_x = 1.92 \times 10^{12} \frac{\text{m}}{\text{s}^2}$. If they move through a displacement $x - x_0 = 1.00 \text{ cm}$, then one of our favorite equations of one-dimensional kinematics gives us:

$$v_x^2 = v_{x0}^2 + 2a_x(x - x_0) = 0^2 + 2(1.92 \times 10^{12} \frac{\text{m}}{\text{s}^2})(1.00 \times 10^{-2} \text{ m}) = 3.83 \times 10^{10} \frac{\text{m}^2}{\text{s}^2}$$

so that the final velocity (and speed) is

$$v_x = 1.96 \times 10^5 \frac{\text{m}}{\text{s}}$$

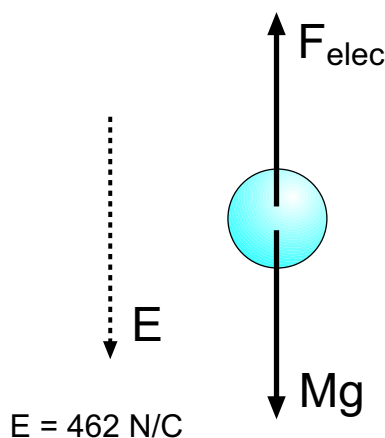


Figure 2.12: Forces acting on the water drop in Example 12.

12. A spherical water drop $1.20\ \mu\text{m}$ in diameter is suspended in calm air owing to a downward-directed atmospheric electric field $E = 462\ \frac{\text{N}}{\text{C}}$. (a) What is the weight of the drop? (b) How many excess electrons does it have? [HRW5 23-49]

(a) Not much physics for this part. The volume of the drop is

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi(1.20 \times 10^{-6})\text{ m}^3 = 9.05 \times 10^{-19}\text{ m}^3$$

Assuming that the density of the drop is the usual density of water, $\rho = 1.00 \times 10^3\ \frac{\text{kg}}{\text{m}^3}$, we can get the mass of the drop from $M = \rho V$. Then the *weight* of the drop is

$$W = Mg = \rho Vg = (1.00 \times 10^3\ \frac{\text{kg}}{\text{m}^3})(9.05 \times 10^{-19}\text{ m}^3)(9.80\ \frac{\text{m}}{\text{s}^2}) = 8.87 \times 10^{-15}\text{ N}$$

(b) The forces acting on the water drop are as shown in Fig. 2.12, namely gravity with magnitude Mg directed downward and the electric force with magnitude $F_{\text{elec}} = |q|E$, directed upward. Here, E is the magnitude of the electric field and $|q|$ is the magnitude of the charge on the drop. The net force on the drop is zero, and so this allows us to solve for $|q|$:

$$F_{\text{elec}} = |q|E = Mg \quad \implies \quad |q| = \frac{Mg}{E}$$

Plug in the weight $W = Mg$ found in part (a) and the given value of the E :

$$|q| = \frac{Mg}{E} = \frac{(8.87 \times 10^{-15}\text{ N})}{(462\ \frac{\text{N}}{\text{C}})} = 1.92 \times 10^{-17}\text{ C}$$

We know that the drop has a *negative* charge (electric field points down, but the electric *force* points *up*) so that the charge on the drop is

$$q = -1.92 \times 10^{-17}\text{ C} .$$

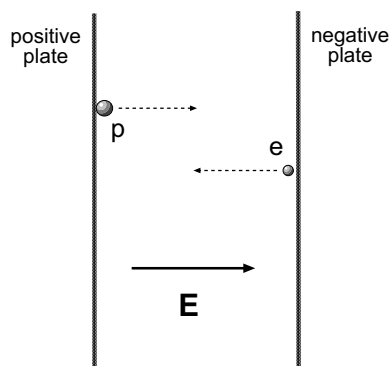


Figure 2.13: Proton and electron are released at the same time and move in opposite directions in a uniform electric field, as given in Example 13.



Figure 2.14: Coordinates for the motion of proton and electron in Example 13.

This negative charge comes from an accumulation of electrons on the water drop. The charge of *one* electron is $q_e = -1.60 \times 10^{-19} \text{ C}$. So the number of electrons on the drop must be

$$N = \frac{(-1.92 \times 10^{-17} \text{ C})}{(-1.60 \times 10^{-19} \text{ C})} = 120$$

The drop has 120 excess electrons.

13. Two large parallel copper plates are 5.0 cm apart and have a uniform electric field between them as depicted in Fig. 2.13. An electron is released from the negative plate at the same time that a proton is released from the positive plate. Neglect the force of the particles on each other and find their distance from the positive plate when they pass each other. (Does it surprise you that you need not know the electric field to solve this problem?) [HRW6 23-41]

We organize our work by setting up coordinates; we suppose that the proton and electron both move along the x axis, as shown in Fig. 2.14 (though we ignore the force they exert on each other). If the distance between the plates is $D = 5.0 \text{ cm}$ then the proton starts off at $x = 0$ and the electron starts off at $x = D$. The proton will accelerate in the $+x$ direction and the electron will accelerate in the $-x$ direction.

The charge of the proton is $+e$. The electric field between the plates points in the $+x$ direction and has magnitude E . Then the force on the proton has magnitude eE and by

Newton's second law the magnitude of the proton's acceleration is $\frac{eE}{m_p}$, where m_p is the mass of the proton. Then the x -acceleration of the proton is

$$a_{x,p} = \frac{eE}{m_p}$$

and since the proton starts from rest, its position is given by

$$x_p = \frac{1}{2} \frac{eE}{m_p} t^2 \quad (2.16)$$

The charge of the electron is $-e$. Then the force on the electron will have magnitude eE and point in the $-x$ direction. The magnitude of the electron's acceleration is $\frac{eE}{m_e}$, where m_e is the mass of the electron, but the electron's acceleration points in the $-x$ direction. Then the x -acceleration of the electron is

$$a_{x,e} = -\frac{eE}{m_e}$$

and since the electron starts from rest and is initially at $x = D$, its position is given by

$$x_e = D - \frac{1}{2} \frac{eE}{m_e} t^2 \quad (2.17)$$

Clearly, there is some time at which x_p and x_e are equal. This happens when

$$\frac{1}{2} \frac{eE}{m_p} t^2 = D - \frac{1}{2} \frac{eE}{m_e} t^2$$

A little bit of algebra will allow us to solve for t . Regroup some terms:

$$\begin{aligned} \frac{1}{2} \frac{eE}{m_p} t^2 + \frac{1}{2} \frac{eE}{m_e} t^2 &= D \\ \frac{1}{2} eE t^2 \left(\frac{1}{m_p} + \frac{1}{m_e} \right) &= D \\ t^2 &= \frac{2D}{eE} \left(\frac{1}{m_p} + \frac{1}{m_e} \right)^{-1} \end{aligned} \quad (2.18)$$

Taking the square root to get t is not necessary because we want to plug t back into either one of the equations for the coordinates to find the value of x at which the meeting occurred, and both of those equations contain t^2 . Putting 2.18 into 2.16 we find:

$$\begin{aligned} x_{\text{meet}} &= \frac{1}{2} \frac{eE}{m_p} t_{\text{meet}}^2 \\ &= \frac{1}{2} \frac{eE}{m_p} \frac{2D}{eE} \left(\frac{1}{m_p} + \frac{1}{m_e} \right)^{-1} \end{aligned}$$

Then we keep doing algebra to get a beautiful, simple form!

$$\begin{aligned}
 x_{\text{meet}} &= \frac{D}{m_p} \left(\frac{1}{m_p} + \frac{1}{m_e} \right)^{-1} \\
 &= \frac{D}{m_p} \left(\frac{m_p m_e}{(m_e + m_p)} \right) \\
 &= \frac{D m_e}{(m_e + m_p)} \tag{2.19}
 \end{aligned}$$

Now just plug numbers into Eq. 2.19. We are given D , and we also need:

$$m_p = 1.67 \times 10^{-27} \text{ kg} \quad m_e = 9.11 \times 10^{-31} \text{ kg}$$

These give:

$$\begin{aligned}
 D &= \frac{(5.0 \times 10^{-2} \text{ m})(9.11 \times 10^{-31} \text{ kg})}{(9.11 \times 10^{-31} \text{ kg} + 1.67 \times 10^{-27} \text{ kg})} \\
 &= 2.73 \times 10^{-5} \text{ m} = 27 \mu\text{m}
 \end{aligned}$$

As for the concluding question...

We note that the final answer did not depend on the value of the electric field strength E (a good thing, since it wasn't given). We can think about why this happened: Since at the meeting place both particles had been traveling for the same amount of time, the distances traveled by each are proportional to their accelerations. If we change *both* accelerations by the same factor, the meeting point will occur at the *same* place because the ratio of distances traveled by each particle is the same. So applying the same scale factor to both a_p and a_e does not change the answer. But changing the value of the electric field *does* apply the same scale factor to both accelerations; since this will not change the answer, we don't expect to see E show up in the expression for x_{meet} .

(That was long-winded... do you have a simpler way to see this?)

14. The electrons in a particle beam each have a kinetic energy of $1.60 \times 10^{-17} \text{ J}$. What are the magnitude and direction of the electric field that will stop these electrons in a distance of 10.0 cm? [Ser4 23-47]

Let's think about the direction first. The acceleration of the electrons must be *directly opposite* their initial (beam) velocities in order for them to come to a halt. So the force on them is also opposite the beam direction. From the *vector* equation $\mathbf{F} = q\mathbf{E}$ we see that if the charge q is negative—as it is for an electron—then the force and electric field have opposite directions. So the electric field must point in the *same* direction as the initial velocities of the electrons (the beam direction). See Fig. 2.15.

It is easiest to use the work-energy theorem to solve the problem. Recall:

$$W_{\text{net}} = \Delta K$$

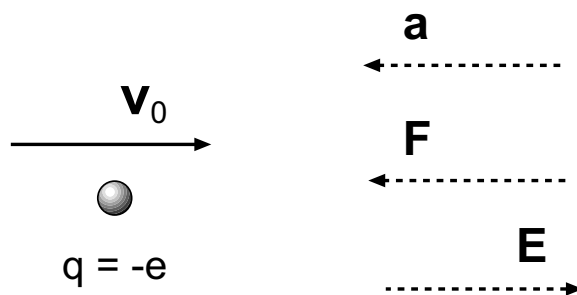


Figure 2.15: Directions for the acceleration, force and (uniform) electric field for Example 14.

As the electron slows to a halt, its change in kinetic energy is

$$\Delta K = K_f - K_i = 0 - (1.60 \times 10^{-17} \text{ J}) = -1.60 \times 10^{-17} \text{ J}$$

Suppose the electric force on the electron has magnitude F . The electron moves a distance $s = 10.0 \text{ cm}$ *opposite* the direction of the force so that the work done is

$$W = -Fd = -F(10.0 \times 10^{-2} \text{ m})$$

which is also the *net* work done. The work–energy theorem says that these are equal, so:

$$-F(10.0 \times 10^{-2} \text{ m}) = -1.60 \times 10^{-17} \text{ J}$$

Solve for F :

$$F = \frac{(1.60 \times 10^{-17} \text{ J})}{(10.0 \times 10^{-2} \text{ m})} = 1.60 \times 10^{-16} \text{ N}$$

Now since we know the charge of an electron we can find the magnitude of the electric field. Here we have $E = F/|q| = F/e$, so the magnitude of the E field is

$$E = \frac{F}{e} = \frac{(1.60 \times 10^{-16} \text{ N})}{(1.60 \times 10^{-19} \text{ C})} = 1.00 \times 10^3 \frac{\text{N}}{\text{C}}$$

We have now found both the magnitude *and* direction of the E field.

